A NOTE ON THE SYMMETRIC POWERS OF THE STANDARD REPRESENTATION OF S_n

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Abstract

In this paper, we prove that the dimension of the space spanned by the characters of the symmetric powers of the standard n-dimensional representation of S_n is asymptotic to $n^2/2$. This is proved by using generating functions to obtain formulas for upper and lower bounds, both asymptotic to $n^2/2$, for this dimension. In particular, for $n \geq 7$, these characters do not span the full space of class functions on S_n.

Notation

Let $P(n)$ denote the number of (unordered) partitions of n into positive integers, and let ϕ denote the Euler totient function. Let V be the standard n-dimensional representation of S_n, so that $V = \mathbb{C}e_1 \oplus \cdots \oplus \mathbb{C}e_n$ with $\sigma(e_i) = e_{\sigma i}$ for $\sigma \in S_n$. Let $S^N V$ denote the N^{th} symmetric power of V, and let $\chi_N : S_n \to \mathbb{Z}$ denote its character. Finally, let $D(n)$ denote the dimension of the space of class functions on S_n spanned by all the χ_N, $N \geq 0$.

1. Preliminaries

Our aim in this paper is to investigate the numbers $D(n)$. It is a fundamental problem of invariant theory to decompose the character of the symmetric powers of an irreducible representation of a finite group (or more generally a reductive group). A special case with a nice theory is the reflection representation of a finite Coxeter group. This is essentially what we are looking at. (The defining representation of S_n consists of the direct sum of the reflection representation and the trivial representation. This trivial summand has no significant effect on the theory.) In this context

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it seems natural to ask: what is the dimension of the space spanned by the symmetric powers? Moreover, decomposing the symmetric powers of the character of an irreducible representation of S_n is an example of the operation of *inner plethysm* [1, Exer. 7.74], so we are also obtaining some new information related to this operation.

We begin with:

Lemma 1.1. *Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be a partition of n (which we denote by $\lambda \vdash n$), and suppose $\sigma \in S_n$ is a λ-cycle. Then $\chi_N(\sigma)$ is equal to the number of solutions $(x_1, \dots, x_k)$ in nonnegative integers to the equation $\lambda_1 x_1 + \cdots + \lambda_k x_k = N$.*

Proof. Suppose without loss of generality that $\sigma = (1\ 2\ \cdots\ \lambda_1)(\lambda_1+1\ \cdots\ \lambda_1+\lambda_2)\cdots(\lambda_1+\cdots+\lambda_{k-1}+1\ \cdots\ n)$. Consider a basis vector $e_1^{\otimes c_1} \otimes \cdots \otimes e_n^{\otimes c_n}$ of $S^N V$, so that $c_1 + \cdots + c_n = N$ with each $c_i \geq 0$. This vector is fixed by σ if and only if $c_1 = \cdots = c_{\lambda_1}$, $c_{\lambda_1+1} = \cdots = c_{\lambda_1+\lambda_2}$ and so on. Since $\chi_N(\sigma)$ equals the number of basis vectors fixed by σ, the lemma follows. □

It seems difficult to work directly with the χ_N's; fortunately, it is not too hard to restate the problem in more concrete terms. Given a partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n, define

$$f_\lambda(q) = \frac{1}{(1-q^{\lambda_1})\cdots(1-q^{\lambda_k})}. \tag{1}$$

Next, define $F_n \subset \mathbb{C}[[q]]$ to be the complex vector space spanned by all of these $f_\lambda(q)$'s. We have:

Proposition 1.2. $\dim F_n = D(n)$.

Proof. Consider the table of the characters χ_N; we are interested in the dimension of the row-span of this table. Since the dimension of the row-span of a matrix is equal to the dimension of its column-span, we can equally well study the dimension of the space spanned by the columns of the table. By the preceeding lemma, the N^{th} entry of the column corresponding to the λ-cycles is equal to the number of nonnegative integer solutions to the equation $\lambda_1 x_1 + \cdots + \lambda_k x_k = N$. Consequently, one easily verifies that $f_\lambda(q)$ is the generating function for the entries of the column corresponding to the λ-cycles. The dimension of the column-span of our table is therefore equal to $\dim F_n$, and the proposition is proved.

□

2. Upper Bounds on $D(n)$

Our basic strategy for computing upper bounds for $\dim F_n$ is to put all the generating functions $f_\lambda(q)$ over a common denominator; then the dimension of their span is bounded above by 1 plus the degree of their numerators. For example, one can see without much difficulty that $(1-q)(1-q^2)\cdots(1-q^n)$ is the least common multiple of the denominators of the $f_\lambda(q)$'s. Putting all of the $f_\lambda(q)$'s over this common

denominator, their numerators then have degree $n(n+1)/2 - n$, which proves

$$D(n) \leq \frac{n(n-1)}{2} + 1. \tag{2}$$

By modifying this strategy carefully, it is possible to find a somewhat better bound. Observe that the denominator of each of our f_λ's is (up to sign change) a product of cyclotomic polynomials. In fact, the power of the j^{th} cyclotomic polynomial $\Phi_j(q)$ dividing the denominator of $f_\lambda(q)$ is precisely equal to the number of λ_i's which are divisible by j. It follows that $\Phi_j(q)$ divides the denominator of $f_\lambda(q)$ at most $\left\lfloor \frac{n}{j} \right\rfloor$ times, and the partitions λ for which this upper bound is achieved are precisely the $P\left(n - j\left\lfloor \frac{n}{j} \right\rfloor\right)$ partitions of n which contain $\left\lfloor \frac{n}{j} \right\rfloor$ copies of j. Let S_j be the collection of f_λ's corresponding to these $P\left(n - j\left\lfloor \frac{n}{j} \right\rfloor\right)$ partitions. One sees immediately that the dimension of the space spanned by the functions in S_j is just $D\left(n - j\left\lfloor \frac{n}{j} \right\rfloor\right)$: in fact, the functions in this space are exactly $1/(1-q^j)^{\left\lfloor \frac{n}{j} \right\rfloor}$ times the functions in $F_{n-j\left\lfloor \frac{n}{j} \right\rfloor}$.

Now the power of $\Phi_j(q)$ in the least common multiple of the denominators of all of the $f_\lambda(q)$'s *excluding those in* S_j is only $\left\lfloor \frac{n}{j} \right\rfloor - 1$, so the degree of this common denominator is only $n(n+1)/2 - \phi(j)$. Therefore, as in the first paragraph of this section, the dimension of the space spanned by all of the f_λ's except those in S_j is at most $n(n-1)/2 + 1 - \phi(j)$; since the dimension spanned by the functions in S_j is $D\left(n - j\left\lfloor \frac{n}{j} \right\rfloor\right)$, we have proved the upper bound

$$D(n) \leq \frac{n(n-1)}{2} + 1 - \phi(j) + D\left(n - j\left\lfloor \frac{n}{j} \right\rfloor\right).$$

If it happens that $D\left(n - j\left\lfloor \frac{n}{j} \right\rfloor\right) < \phi(j)$, then this upper bound is an improvement on our original upper bound. If we repeat this process, this time simultaneously excluding the sets S_j for *all* of the j's which gave us an improved upper bound in the above argument, we find that we have proved:

Proposition 2.1.

$$D(n) \leq \frac{n(n-1)}{2} + 1 - \sum_{j=1}^{n} \max\left(0, \phi(j) - D\left(n - j\left\lfloor \frac{n}{j} \right\rfloor\right)\right).$$

Finally, we obtain an upper bound for $D(n)$ which does not depend on other values of $D(\cdot)$:

Corollary 2.2. *Recursively define* $U(0) = 1$ *and*

$$U(n) = \frac{n(n-1)}{2} + 1 - \sum_{j=1}^{n} \max\left(0, \phi(j) - U\left(n - j\left\lfloor \frac{n}{j} \right\rfloor\right)\right).$$

Then $D(n) \leq U(n)$.

Proof. We proceed by induction on n. Equality certainly holds for $n = 0$. For larger n, the inductive hypothesis shows that $D\left(n - j\left\lfloor \frac{n}{j}\right\rfloor\right) \leq U\left(n - j\left\lfloor \frac{n}{j}\right\rfloor\right)$ when $j > 0$, and so

$$\begin{aligned} D(n) &\leq \frac{n(n-1)}{2} + 1 - \sum_{j=1}^{n} \max\left(0, \phi(j) - D\left(n - j\left\lfloor \frac{n}{j}\right\rfloor\right)\right) \\ &\leq \frac{n(n-1)}{2} + 1 - \sum_{j=1}^{n} \max\left(0, \phi(j) - U\left(n - j\left\lfloor \frac{n}{j}\right\rfloor\right)\right) \\ &= U(n). \end{aligned}$$

□

Below is a table of values of $D(n)$ and $U(n)$ for $n \leq 23$, calculated in Maple, with $P(n)$ and our first estimate $\frac{n(n-1)}{2} + 1$ provided for contrast. Note that in the range $1 \leq n \leq 23$, we have $D(n) = U(n)$ except for $n = 19, 20$, when $U(n) - D(n) = 1$. Is it true, for instance, that

$$-D(n) + \frac{n(n-1)}{2} + 1 - \sum_{j=1}^{n} \max\left(0, \phi(j) - D\left(n - j\left\lfloor \frac{n}{j}\right\rfloor\right)\right)$$

is bounded as $n \to \infty$?

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$D(n)$	1	2	3	5	7	11	13	19	23	29	35	45	51	62
$U(n)$	1	2	3	5	7	11	13	19	23	29	35	45	51	62
$n(n-1)/2+1$	1	2	4	7	11	16	22	29	37	46	56	67	79	92
$P(n)$	1	2	3	5	7	11	15	22	30	42	56	77	101	135

n	15	16	17	18	19	20	21	22	23
$D(n)$	69	79	90	106	118	134	146	161	176
$U(n)$	69	79	90	106	119	135	146	161	176
$n(n-1)/2+1$	106	121	137	154	172	191	211	232	254
$P(n)$	176	231	297	385	490	627	792	1002	1255

TABLE 1. Values of $D(n)$, $U(n)$, $n(n-1)/2+1$, $P(n)$ for small n

Example 1. The first dimension where $D(n) < P(n)$ is $n = 7$, and it is easy then to show that $D(n) < P(n)$ for all $n \geq 7$. The difference $P(7) - D(7) = 2$ arises from the following two relations:

$$\frac{4}{(1-x^2)^2(1-x)^3} = \frac{3}{(1-x^3)(1-x)^4} + \frac{1}{(1-x^3)(1-x^2)^2}$$

and

$$\frac{3}{(1-x^3)(1-x^2)(1-x)^2} = \frac{2}{(1-x^4)(1-x)^3} + \frac{1}{(1-x^4)(1-x^3)}.$$

The first relation, for example, says that if χ is a linear combination of χ_N's, then

$$4 \cdot \chi((2,2)\text{-cycle}) = 3 \cdot \chi(3\text{-cycle}) + \chi((3,2,2)\text{-cycle}).$$

Alternately, it tells us that for any $N \geq 0$, four times the number of nonnegative integral solutions to $2x_1 + 2x_2 + x_3 + x_4 + x_5 = N$ is equal to three times the number of such solutions to $3x_1 + x_2 + x_3 + x_4 + x_5 = N$ plus the number of such solutions to $3x_1 + 2x_2 + 2x_3 = N$.

3. Lower Bounds on $D(n)$

Let $\lambda = (\lambda_1, \dots, \lambda_k) \vdash n$. The rational function $f_\lambda(q)$ of equation (1) can be written as

$$f_\lambda(q) = p_\lambda(1, q, q^2, \dots),$$

where p_λ denotes a power sum symmetric function. (See [1, Ch. 7] for the necessary background on symmetric functions.) Since the p_λ for $\lambda \vdash n$ form a basis for the vector space (say over $\mathbb{C}$) Λ^n of all homogeneous symmetric functions of degree n [1, Cor. 7.7.2], it follows that if $\{u_\lambda\}_{\lambda \vdash n}$ is any basis for Λ^n then

$$D(n) = \dim \operatorname{span}_{\mathbb{C}} \{u_\lambda(1, q, q^2, \dots) \,:\, \lambda \vdash n\}.$$

In particular, let $u_\lambda = e_\lambda$, the elementary symmetric function indexed by λ. Define

$$d(\lambda) = \sum_i \binom{\lambda_i}{2}.$$

According to [1, Prop. 7.8.3], we have

$$e_\lambda(1, q, q^2, \dots) = \frac{q^{d(\lambda)}}{\prod_i (1-q)(1-q^2)\cdots(1-q^{\lambda_i})}.$$

Since power series of different degrees (where the *degree* of a power series is the exponent of its first nonzero term) are linearly independent, we obtain from Proposition 1.2 the following result.

Proposition 3.1. *Let $E(n)$ denote the number of distinct integers $d(\lambda)$, where λ ranges over all partitions of n. Then $D(n) \geq E(n)$.*

Note. We could also use the basis s_λ of Schur functions instead of e_λ, since by [1, Cor. 7.21.3] the degree of the power series $s_\lambda(1, q, q^2, \dots)$ is $d(\lambda')$, where λ' denotes the conjugate partition to λ.

Define $G(n) + 1$ to be the least positive integer that cannot be written in the form $\sum_i \binom{\lambda_i}{2}$, where $\lambda \vdash n$. Thus all integers $1, 2, \dots, G(n)$ can be so represented, so $D(n) \geq E(n) \geq G(n)$. We can obtain a relatively tractable lower bound for $G(n)$, as follows. For a positive integer m, write (uniquely)

$$m = \binom{k_1}{2} + \binom{k_2}{2} + \cdots + \binom{k_r}{2}, \tag{3}$$

where $k_1 \geq k_2 \geq \cdots \geq k_r \geq 2$ and $k_1, k_2, \ldots$ are chosen successively as large as possible so that

$$m - \binom{k_1}{2} - \binom{k_2}{2} - \cdots - \binom{k_i}{2} \geq 0$$

for all $1 \leq i \leq r$. For instance, $26 = \binom{7}{2} + \binom{3}{2} + \binom{2}{2} + \binom{2}{2}$. Define $\nu(m) = k_1 + k_2 + \cdots + k_r$. Suppose that $\nu(m) \leq n$ for all $m \leq N$. Then if $m \leq N$ we can write $m = \binom{k_1}{2} + \cdots + \binom{k_r}{2}$ so that $k_1 + \cdots + k_r \leq n$. Hence if $\lambda = \left(k_1, \ldots, k_r, 1^{n-\sum k_i}\right)$ (where 1^s denotes s parts equal to 1), then λ is a partition of n for which $\sum_i \binom{\lambda_i}{2} = m$. It follows that if $\nu(m) \leq n$ for all $m \leq N$ then $G(n) \geq N$. Hence if we define $H(n)$ to be the largest integer N for which $\nu(m) \leq n$ whenever $m \leq N$, then we have established the string of inequalities

$$D(n) \geq E(n) \geq G(n) \geq H(n). \tag{4}$$

Here is a table of values of these numbers for $1 \leq n \leq 23$. Note that $D(n)$ appears to be close to $E(n+1)$. We don't have any theoretical explanation of this observation.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$D(n)$	1	2	3	5	7	11	13	19	23	29	35	45	51	62
$E(n)$	1	2	3	5	7	9	13	18	21	27	34	39	46	54
$G(n)$	0	1	1	3	4	4	7	13	13	18	25	32	32	32
$H(n)$	0	1	1	3	4	4	7	11	13	18	19	19	25	32

n	15	16	17	18	19	20	21	22	23
$D(n)$	69	79	90	106	118	134	146	161	176
$E(n)$	61	72	83	92	106	118	130	145	162
$G(n)$	40	49	52	62	73	85	102	112	127
$H(n)$	40	43	52	62	73	85	89	102	116

TABLE 2. Values of $D(n)$, $E(n)$, $G(n)$, $H(n)$ for small n

Proposition 3.2. *We have*

$$\nu(m) \leq \sqrt{2m} + 3m^{1/4} \tag{5}$$

for all $m \geq 405$.

Proof. The proof is by induction on m. It can be checked with a computer that equation (5) is true for $405 \leq m \leq 50000$. Now assume that $M > 50000$ and that (5) holds for $405 \leq m < M$. Let $p = p_M$ be the unique positive integer satisfying

$$\binom{p}{2} \leq M < \binom{p+1}{2}.$$

Thus p is just the integer k_1 of equation (3). Explicitly we have

$$p_M = \left\lfloor \frac{1 + \sqrt{8M+1}}{2} \right\rfloor.$$

By the definition of $\nu(M)$ we have

$$\nu(M) = p_M + \nu\left(M - \binom{p_M}{2}\right).$$

It can be checked that the maximum value of $\nu(m)$ for $m < 405$ is $\nu(404) = 42$. Set $q_M = (1 + \sqrt{8M+1})/2$. Since $M - \binom{p_M}{2} \leq p_M \leq q_M$, by the induction hypothesis we have

$$\nu(M) \leq q_M + \max(42, \sqrt{2q_M} + 3q_M^{1/4}).$$

It is routine to check that when $M > 50000$ the right hand side is less than $\sqrt{2M} + 3M^{1/4}$, and the proof follows. □

Proposition 3.3. *There exists a constant $c > 0$ such that*

$$H(n) \geq \frac{n^2}{2} - cn^{3/2}$$

for all $n \geq 1$.

Proof. From the definition of $H(n)$ and Proposition 3.2 (and the fact that the right-hand side of equation (5) is increasing), along with the inquality $\nu(m) \leq 42 = \lceil\sqrt{2 \cdot 405} + 3 \cdot 405^{1/4}\rceil$ for $m \leq 404$, it follows that

$$H\left(\lceil\sqrt{2m} + 3m^{1/4}\rceil\right) \geq m$$

for $m > 404$. For n sufficiently large, we can evidently choose m such that $n = \lceil\sqrt{2m} + 3m^{1/4}\rceil$, so $H(n) \geq m$. Since $\sqrt{2m} + 3m^{1/4} + 1 > n$, an application of the quadratic formula (again for n sufficiently large) shows

$$m^{1/4} \geq \frac{-3 + \sqrt{9 + 4\sqrt{2}(n-1)}}{2\sqrt{2}},$$

from which the result follows without difficulty.

□

Since we have established both upper bounds (equation (2)) and lower bounds (equation (4) and Proposition 3.3) for $D(n)$ asymptotic to $n^2/2$, we obtain the following corollary.

Corollary 3.4. *There holds the asymptotic formula $D(n) \sim \frac{1}{2}n^2$.*

References

[1] R. Stanley, *Enumerative Combinatorics*, vol. 2, Cambridge University Press, New York/Cambridge, 1999.